Theoretic Limits on the Equation of State Parameter of Quintessence

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The value of scalar field coupled to gravity should be less than the Planck scale in the consistent theory of quantum gravity. It provides a theoretic constraint on the equation of state parameter for the quintessence. In some cases our theoretic constraints are more stringent than the constraints from the present experiments.

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The accelerating cosmic expansion is first inferred from the observations of distant type Ia supernovae [1]. It indicates unexpected gravitational physics attributed to the dominating presence of a dark energy with negative pressure. Some other independent observations, such the cosmic microwave background radiation (CMBR) and Sloan Digital Sky Survey (SDSS), also strongly favor dark energy as the dominant component in the present mass-energy budget of the Universe.

A simply candidate for the dark energy is Einstein's famous cosmological constant ρ_Λ. Nowadays it is still consistent with all of the observations. See the recent analysis of the experiments in [2, 3]. The action for the description of the Universe takes the form

$$S=\frac{M_p^2}{2}\int d^4x\sqrt{-g}R+\int d^4x\sqrt{-g}\mathcal{L}(\phi_i,\psi_j,A_\mu,...), \tag{1}$$

where M_p is the reduced Planck scale, ϕ_i is the scalar field, ψ_j is the Fermionic field, A_μ is the gauge field and so on. However anything that contributes to the energy density of the vacuum acts just like a cosmological constant. If we treat these quantum fields independently, there is a zero point energy coupled to gravity. Summing the zero point energies of all normal modes of some field of mass m up to a wave number cutoff $\Lambda \gg m$ yields a vacuum energy density

$$\langle\rho\rangle=\int_0^\Lambda \frac{4\pi k^2 dk}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2}\simeq\frac{\Lambda^4}{16\pi^2}. \tag{2}$$

If we believe that the Planck scale is a natural cutoff for the quantum field theories, $\langle\rho\rangle = M_p^4/(16\pi^2)$ is much greater than the observed value of the energy density of dark energy $\rho_D = 10^{-123}M_p^4$. The energy scale for the local effective field theory related to the cosmological constant is roughly 10^{-3} eV. The puzzle is why the vacuum energy is so small after including all of these contributions. Another problem is why the energy density of the dark energy is comparable to the matter energy density now (cosmic coincidence problem). For a classic review see [4], for a recent nice review see [5], and for a recent discussion see [6].

The theory of quantum gravity is needed to solve the cosmological constant problem. String theory appears to be a consistent and well-defined theory of quantum gravity. In [7] Arkani-Hamed et al. suggest that the gravity and other quantum field theory cannot be treated independently in quantum gravity. For instance, a new intrinsic UV cutoff $\Lambda = gM_p$ for the U(1) gauge theory with coupling g coupled to gravity arises in four-dimensional Minkowski spacetime. They also check this conjecture in a few examples in string theory. The other concerning on this conjecture is [8]. This conjecture is generalized to the asymptotical de Sitter spacetime [9]. The Hubble parameter H plays the role as the IR cutoff for the effective field theory. Requiring that the IR cutoff be less than the UV cutoff leads to an upper bound on the cosmological constant $\rho_\Lambda \le g^2M_p^4$ [9]. This conjecture has a simple explanation in string theory: the string length $\sqrt{\alpha'}$ should be shorter than the size of the cosmic horizon. It can be easily checked in the brane world scenario. See [9] in detail. If there is a U(1) gauge theory with incredibly small coupling $g \sim 10^{-60}$ in our universe, we can understand why the cosmological constant is so small. Similarly a conjecture for the $\lambda\phi^4$ theory is proposed as $\Lambda = \lambda^{1/2}M_p$ in the Minkowski spacetime and $\rho_\Lambda \le \lambda M_p^4$ in the asymptotical de Sitter space [10]. This conjecture implies that the value of ϕ cannot be larger than the Planck scale M_p and the chaotic inflation cannot be achieved. However this conjecture is limited to $\lambda\phi^4$ theory. We propose a general conjecture that the description of the scalar field theory breaks down in the over-Planckian field space in [11] where several examples in string theory are discussed. For the other arguments in string theory to support this conjecture see [12].

If the observed dark energy is really a small positive cosmological constant the ultimate future of our universe will be eternal de Sitter space. This would mean not that the future is totally empty space, but that the would will have all the features of an isolated finite thermal cavity with finite temperature and entropy $S_{dS}=\frac{\text{Horizon Area}}{4G}$ [13]. The entropy reaches it maximum value and the second law forbids any further interesting history. But on a sufficiently long time scale, large fluctuations will occur. For de Sitter space the Poincare recurrences generally occur on a time scale exponentially large in the thermal entropy of the system $e^{S_{dS}}$ [14]. Recently the authors

in [15] propose that in the de Sitter space the description of the local effective field theory breaks down after a time scale S_{dS} which is much shorter than the recurrence time. Another trouble with the positive cosmological constant is that it does not appear possible to define precise observables, at least none that can be measured by an observer in the spacetime [16].

Another source for an appropriate dark energy component is a single slow-rolling scalar field called quintessence [17]. In an expanding universe, a spatially homogeneous canonical scalar field with potential $V(\phi)$ and minimal coupling to gravity obeys

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{3}$$

where the dot and prime denote the derivative with respect to the cosmic time and quintessence ϕ respectively. The energy density is $\rho_Q = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, and the pressure is $p_Q = \frac{1}{2}\dot{\phi}^2 - V(\phi)$, implying an equation of state parameter

$$w \equiv \frac{p_Q}{\rho_Q} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}, \tag{4}$$

which generally varies with time. The range for the equation of state parameter for the quintessence is $w \in [-1, 1]$. The Hubble parameter H acts as a friction term. When the friction term is large enough, the field is slowly rolls down its potential and $\dot{\phi}^2 \ll V(\phi)$. Now $w \simeq -1$ and the quintessence acts like a cosmological constant.

This picture relies on an application of low-energy effective field theory to the quintessence. So the variation of quintessence should be less than M_p. In this paper, we will investigate the theoretic constraint on the equation of state parameter for the quintessence due to the sub-Planckian excursion in the field space.

For simplicity of calculations we assume spatial flatness which is motivated by theoretical considerations, such as inflation, and observations. Our results can be easily generalized to the case with a spatial curvature. The Hubble parameter is given by

$$H^2 = \frac{\rho_{crit}}{3M_p^2} = \frac{\rho_Q + \rho_m}{3M_p^2}, \tag{5}$$

where $\rho_m(z) = \rho_m^0(1+z)^3$ is the energy density of the dust-like matter in our universe and ρ_m^0 is its energy density at present. Here we normalize $a_0 = 1$ and the scale factor is related to the redshift z by $a = (1+z)^{-1}$. Using eq. (4), we find a relationship between the potential of quintessence and its kinetic energy

$$V(\phi) = \frac{\dot{\phi}^2}{2}\frac{1-w}{1+w}. \tag{6}$$

The energy density takes the form

$$\rho_Q = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \frac{\dot{\phi}^2}{1+w}. \tag{7}$$

In the whole paper, we assume, without loss of generality, $V' < 0$, so that $\dot{\phi} > 0$. Thus eq. (7) reads

$$\dot{\phi} = \sqrt{(1+w)\rho_Q}. \tag{8}$$

Integrating eq. (8), we obtain

$$\begin{aligned} \frac{|\Delta\phi(z)|}{M_p} &= \int_{\phi(z)}^{\phi(0)} d\phi/M_p \\ &= \int_0^z \sqrt{3[1+w(z')]\Omega_Q(z')}\frac{dz'}{1+z'}, \end{aligned} \tag{9}$$

which should be less than 1. Here we use $Hdt = -\frac{dz}{1+z}$. The density parameter for the quintessence is

$$\Omega_Q = \frac{\rho_Q}{\rho_{crit}} = \frac{\rho_Q}{\rho_Q + \rho_m}. \tag{10}$$

The energy conservation implies

$$\dot{\rho}_Q + 3H(\rho_Q + p_Q) = 0. \tag{11}$$

Combining with eq. (4), we solve (11) as

$$\rho_Q(z) = \rho_Q^0 \exp\left(\int_0^z 3\frac{1+w(z')}{1+z'}dz'\right), \tag{12}$$

where ρ_Q^0 is the present energy density of quintessence. Using (12), we obtain

$$\frac{1}{\Omega_Q(z)} = 1 + \frac{\Omega_m^0}{\Omega_Q^0}(1+z)^3 \exp\left(-\int_0^z 3\frac{1+w(z')}{1+z'}dz'\right). \tag{13}$$

For the case with a spatial curvature, we only need to add another term $-\frac{\Omega_k^0}{\Omega_Q^0}(1+z)^2 \exp\left(-\int_0^z 3\frac{1+w(z')}{1+z'}dz'\right)$ on the right hand side of eq. (13). Here we set $\Omega_k^0 = 0$.

Our strategy is using the condition $|\Delta\phi(z)|/M_p < 1$ to constraint the equation of state parameter $w(z)$ for the quintessence. Unfortunately, present dynamical dark energy models in the literatures do not suggest a universal or fundamental parametric form for $w(z)$. For recent review see [18]. We will investigate several typical parameterizations of $w(z)$. There are also strong degeneracies in the effect of $w(z)$ and Ω_m on the expansion history. According to the literatures, we reasonably set $\Omega_Q^0 = 0.72$ and $\Omega_m^0 = 0.28$.

I. $w = w_0$ =const

There are a few models of quintessence that predict an equation of state parameter that is constant, different from the cosmological ($w = -1$). In this case we consider the variation of the quintessence from now to the last scattering ($z_{rec} = 1089$). Requiring $|\Delta\phi(z_{rec})|/M_p < 1$ yields $w = w_0 \leq -0.738$.

In a spatially flat universe, the combination of WMAP and the Supernova Legacy Survey (SNLS) data leads to

a significant constraint on the equation of state parameter for the dark energy $w = -0.967^{+0.073}_{-0.072}$ [3]. The theoretic limit on the equation of state parameter for the quintessence is consistent with the experiments.

II. $w = w_0 + w_1 z$

In this case the equation of state parameter is a linear function of the redshift. This parametrization is studied in [19]. It is a good parametrization at a low redshift. But in this form, $w(z)$ diverges, making it unsuitable at high redshift. As we know, the redshift of the SN sample is less than 2. For the consistence we require $|\Delta\phi(z=2)|/M_p < 1$. The theoretic constraints are $-1 \leq w_0 \leq -0.164$ and $-0.417 \leq w_1 \leq 0.854$. Here we also consider the requirement $w \in [-1, 1]$ for the quintessence. A more explicit result is showed in Fig. 1.

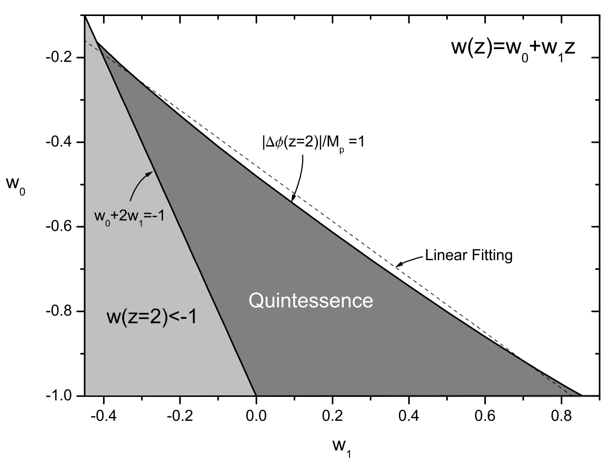

FIG. 1: The gray patch is the prediction of the quintessence. The light gray patch corresponds to $w < -1$. The line with $|\Delta\phi(z=2)|/M_p = 1$ is roughly a straight line which is linearly fitted as $w_0 + 0.657w_1 = -0.456$.

III. $w = w_0 + w_1 \frac{z}{1+z}$

This parametric form is suggested in [20]. It solves the divergence problem in case II and has been widely used in the literatures. Requiring $|\Delta\phi(z_{rec})|/M_p < 1$ yields $-1 \leq w_0 \leq -0.434$ and $-0.564 \leq w_1 \leq 0.498$. See Fig. 2 for the explicit result.

The fitting results for the combination of WMAP3 + SN182 + SDSS + 2dFGRS are $w_0 = -1.149^{+0.543}_{-0.120}$ and $w_1 = 1.017^{+0.146}_{-2.095}$ at 2σ level [21]. The theoretic constraint is consistent with the experiment as well. Taking a closer look, we find the theoretic limit is much more stringent than the present experiments.

The authors in [22] propose that the equation of state parameter has a lower bound, such as $1 + w \geq 0.004$ or 0.01 for two different cases, if the value of the quintessence field is prohibited from attaining values exceeding the Planck scale. Their results contradict with our intuition. The inconsistence in [22] is that

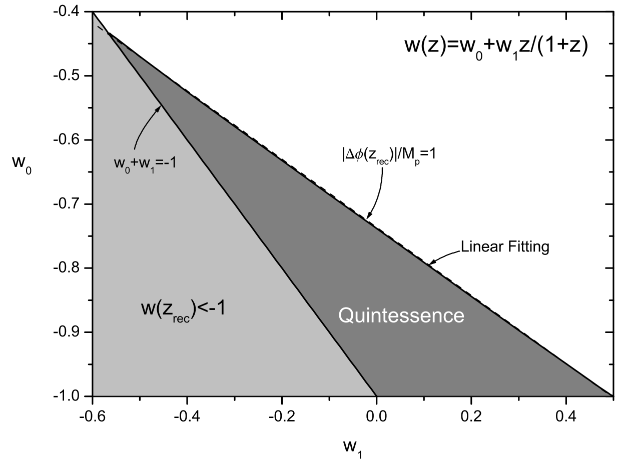

FIG. 2: The gray patch is the prediction of the quintessence. The light gray patch corresponds to $w < -1$. The line with $|\Delta\phi(z=2)|/M_p = 1$ is roughly a straight line which is linearly fitted as $w_0 + 0.532w_1 = -0.736$.

they require $E \equiv |V/V'| < M_p$. Recall what we do for inflation. Similarly we define a slow-roll parameter $\epsilon = \frac{M_p^2}{2}(V'/V)^2 = M_p^2/(2E^2) > 1$, which implies that the quintessence fast rolls down its potential and the accelerating expansion cannot be achieved when this quintessence is the dominant component in the Universe. In fact the absolute value of the scalar field does not make sense because the scalar field can be shifted to be arbitrary value. The reasonable quantity is the value of the scalar field relative to its value in the closed-by stable/metastable vacuum. So the requirement that the variation of the quintessence be less than M_p is more reasonable. The variation of quintessence can be arbitrarily small if its potential is flat enough, and the quintessence just acts like the cosmological constant. The only possible lower bound on w is $1 + w > H_0^2/M_p^2 \sim 10^{-120}$; otherwise, the current inflationary epoch is eternal [15].

To summarize, we investigate the theoretic limits on the equation of state parameter for quintessence through considering that the variation of the canonical quintessence field minimally coupled to gravity is less than the Planck scale. This requirement may arise in the consistent theory of quantum gravity. In this sense our results can be taken as the prediction of quantum gravity. Our theoretic constraints are more stringent than present experiments in some cases. We hope the future observations can confirm the quintessence model or rule it out.

However there is also a new naturalness problem in the quintessence model. In order for the quintessence to be slowly rolling today, the effective mass of quintessence should be smaller than the present Hubble parameter, namely $m_\phi = \sqrt{|V''(\phi_0)|} \leq H_0 \sim 10^{-33}$ eV. This is an incredibly low energy scale compared to the energy scales in particle physics. If so, there must be an unknown

symmetry to protect such a tiny mass of quintessence.

In ten dimensions string theory has no free parameters, but once we compactify, each nonsupersymmetric vacuum will have a different effective cosmological constant [23, 24]. In the reliable set-up in string theory [23], the vacuum with a positive cosmological constant is metastable. The trouble with the eternal de Sitter space we discussed previously seems to be solved. It hints that we should embed the pure gravity with a positive cosmological constant into a bigger theory. Since the cosmological constant is not a dynamical quantity, maybe anthropic principle [25] is needed.

Nowadays the dark energy with $w < -1$, called phantom [26], has not been ruled out by the experiments. If phantom is favored by the future observations, it is a bigger puzzle for the fundamental physics, because the Null energy condition (NEC) is violated, which implies that energy flows faster than the speed of light [27]. The causality is absent as well. A simple realization of phantom is the scalar field with a wrong sign kinetic term [28]. We don't know how to quantize the phantom field at all and the field theory of phantom is ill-defined. To constrain the equation of state parameter for phantom is out of the question. On the other hand, NEC is a crucial assumption in proving the positivity of the ADM mass in asymptotically flat space [29]. The positive energy theorem implies a stable vacuum for gravity and will play a crucial role in quantum gravity.

The cosmological constant problem is still the biggest puzzle in the fundamental physics. We are still far away from the correct answer to it.

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